A model for the motion of a particle in a quantum background

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Abstract

We are studying the dynamics of a one-dimensional field in a non-commutative Euclidean space. The non-commutative space we consider is the one that emerges in the context of three dimensional Euclidean quantum gravity: it is a deformation of the classical Euclidean space $\mathbb{E}^3$ and the Planck length ℓ_P plays the role of the deformation parameter. The field is interpreted as a particle which evolves in a quantum background. When the dynamics of the particle is linear, the resulting motion is similar to the standard motion in the classical space $\mathbb{E}^3$. However, non-linear dynamics on the non-commutative space are different from the corresponding non-linear dynamics on the classical space. These discrepencies are interpreted as "quantum gravity" effects. Finally, we propose a background independent description of the propagation of the particle in the quantum geometry.

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1 Motivations

Loop Quantum Gravity (LQG) [1] has been one of the first theory to turn the question of space-time structure at the Planck scale into a mathematically well-posed problem. This is surely one of the most important and most beautifull achievement of LQG. Indeed, LQG provides a background independent quantization of general relativity where standard geometrical notions, like length, area or volume, become well-defined operators acting on a suitable Hilbert space [2]. Then, the problem of finding how looks space-time at the Planck scale turns out to be the problem of finding the eigeinvalues of these operators. Even if the answer to this last question is still controversary for different reasons (see the recent works ([3]) for instance), it has allowed the possibility to address many other fundamental issues (black hole thermodynamics [6], questions of singularities in classical gravity [7]) that we sometimes did not even know how to tackle before. Even if some of these results are still under discussion, one can claim that LQG offers a simple mathematical framework where one can properly study fundamental aspects of quantum gravity. These last years, we have also seen the emergence of a quantum gravity phenomenology [8] where models have been proposed to describe the low energy regime of quantum gravity. It is nonetheless important to underline that these models, which exhibit very interesting effects, are strongly discussed in the literature and their link with LQG is not clear at all in four dimensions.

In three dimensions (Euclidean signature and no cosmological constant), the situation is somehow simpler: it was argued that quantum gravity effects could be completely recasted into non-commutative effects [9]. In that picture, space-time would become non-commutative at the Planck scale and its "isometry" algebra would be a deformation of the standard classical algebra, known as the Drinfeld double. The deformation parameter is the Newton constant G (or equivalently the Planck length $\ell_P = \hbar G$). Recently, it was precisely shown that this non-commutative space-time, in the Euclidean regime, admits a fuzzy space representation [10]. As a consequence, this model of three dimensional Euclidean quantum space-time has in fact a discrete structure at the Planck scale and the dynamics of fields evolving in such a space become discrete as well. The purpose of this article is to illustrate the effects of the discreteness with some simple but enlighting examples. More precisely, we will consider the dynamics of a one-dimensional field (it depends only on one coordinates out of the three) interpreted as the motion of a particle in a quantum background. At this point, it is important to underline that the system we are studying is a model for the dynamics of particle in a quantum background based on two main asumptions: (i) the quantum background is assumed to be the one that admits the Drinfeld double as its deformed "isometry algebra"; (ii) the one-dimensional field is interpreted as a particle.

The first Section is devoted to briefly recall the construction of the non-commutative space. We start by underlining the importance of the quantum double (or Drinfeld double) $DSU(2)$ in that construction: the non-commutative space is indeed defined as the space that admits $DSU(2)$ as isometry algebra. As the

space is non-commutative, it is indirectly described in terms of its algebra of functions which appears to be the convolution algebra $C(SU(2))^*$ of distributions on the group $SU(2)$. Using harmonic analysis, we show that $C(SU(2))^*$ is isomorphic to the space $\oplus_n \mathrm{Mat}_n(\mathbb{C})$ where $\mathrm{Mat}_n(\mathbb{C})$ are the set of complex matrices of dimension n: this makes clear that the non-commutative space is fuzzy. To make concrete that $C(SU(2))^*$ is a deformation of the algebra $C(\mathbb{E}^3)$ of functions on $\mathbb{E}^3$, we exhibit a link between these two spaces. More precisely, in this article, we restrict our study to the space of functions $C(SU(2))$ and show that it is isomorphic to the direct sum $C_{B_{\ell_P}}(\mathbb{E}^3) \oplus C_{B_{\ell_P}}(\mathbb{E}^3)$ where $C_{B_{\ell_P}}(\mathbb{E}^3)$ is the sub-space of $C(\mathbb{E}^3)$ of functions whose spectrum belongs to the open sphere B_{ℓ_P} of radius ℓ_P^{-1}. The general result for $C(SU(2))^*$ is given in [10]. Thus, any function $\phi \in C(SU(2))$ can be equivalently described by a matrix $\hat{\Phi}$ or by a pair of continuous functions $\Phi_+ \oplus \Phi_-$. We define an integral and derivative operators on $C(SU(2))^*$ which allows to write an action for a scalar field on the non-commutative space. We finish the first Section with a study of the free action for the scalar field. In the second Section, we focus on the dynamics of a one-dimensional field which is interpreted as a particle evolving in a given potential along one "time" direction: equations of motion are written, solutions are found and discussed. In particular, the field admits two components $\Phi_\pm(t)$ when written in the continuous representation: if Φ_+ described the motion of a particle then, in some generic cases, Φ_- is the backward motion in the sense that $\Phi_+(t) = \Phi_-(-t)$. Thus, there is a kind of miror symmetry between the two components. We show that, in the case of a (free) quadratic potentiel, solutions are similar to standard classical ones. Important differences occur when one considers non-linear interactions: the trajectories of a self-interacting particle in a classical or in a quantum background are different. We finish the Section by proposing a background independent interpretation of the propagation in the fuzzy space. We finally conclude with some discussions and perspectives.

2   Quantum geometry as a fuzzy space

It has been argued that the quantum dynamics of a scalar field coupled to three dimensional Euclidean gravity is "equivalent" to the dynamics of a scalar field (with no gravity at all) evolving in a non-commutative three dimensional space. This result has originally been illustrated in the context of covariant spin-foam models coupled to massive spinless particles [9]; it was then recovered in the canonical LQG point of view [10, 11] where the non-commutative space is constructed such that it admits the Drinfeld double $DSU(2)$ as its isometry algebra. In fact, $DSU(2)$ is a deformation of the (group algebra of the) classical Lie group $ISU(2)$.

If one assumes that $DSU(2)$ is effectively the isometry algebra of space at the Planck scale, then the construction of the quantum geometry at the Planck scale is very similar to constructions of model spaces in standard classical geometry. Indeed, in the classical situation, a model space is defined by a coset G/H where G is the isometry (Lie) group and H a subgroup. One can easily adpat this construction to the quantum case, the main important difference being that the resulting coset

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is no longer a manifold. It is implicitly defined by its space of functions, denoted generically C in the sequel, which is endowed with an algebra structure. This algebra contains all the geometrical informations of the non-commutative space.

In our specific case, we showed [10] that C is the algebra of distributions on the Lie group $SU(2)$ endowed with the convolution product. This is a result of the Hopf algebra duality which allows to define in a canonical way an algebraic structure to a (commutative or non-commutative) geometry when its (classical or quantum) symmetry algebra admits a Hopf algebra structure (in fact we just need a co-product). However, this description is rather theoritical and it is necessary to know how to explicitly get the geometrical informations out of it. This has been done in a companion paper [10] and this section aims precisely at recalling some of the results obtained in this paper. First, we show that the quantum geometry described by C is fuzzy in the sense that C is isomorphic to an algebra of complex matrices; then, we exhibit a (non-trivial) link between C and the space $C(\mathbb{E}^3)$ of functions on the classical space $\mathbb{E}^3$; we finish with some properties concerning differential calculus on C, namely we define an integration on C and derivative operators which are necessary to construct an action for non-commutative fields. Finally, we write an action for a non-commutative scalar field and we study, as an example, the case where the field is free.

2.1 Construction of the non-commutative space

For pedagogical purposes, let us start by presenting briefly how the construction works in the classical case before going to the quantum case. In the classical context, C is the pointwise algebra of functions on the classical Euclidean three-dimensional manifold $\mathbb{E}^3$ and our problem consists in constructing C starting from its isometry group algebra $\mathbb{C}[ISU(2)]$ where $ISU(2) = SU(2) \ltimes \mathbb{R}^3$ is the Euclidean group. We consider the group algebra instead of the group itself to be closer to the quantum case (we will present in the sequel).

The solution is simple. We start by introducing the space of $SU(2)$-invariant linear forms on $\mathbb{C}[ISU(2)]$ which is, by definition, the space $C(\mathbb{R}^3)^*$ of distributions on $\mathbb{R}^3$. The product $\circ$ between two such distributions f_1 and f_2 is defined from the grouplike coproduct Δ on $\mathbb{C}[ISU(2)]$ using the Hopf algebra duality principle as follows:

$$f_1 \circ f_2(a) \;=\; (f_1 \otimes f_2)\Delta(a) \quad \text{for any } a \in ISU(2). \tag{1}$$

Doing so, we obtain, after some trivial calculations, that $\circ$ is the standard convolution product in $\mathbb{R}^3$ and then we have constructed the convolution algebra of distributions on $\mathbb{R}^3$. Finally, the algebra C is easily obtained from $C(\mathbb{R}^3)^*$ performing a standard Fourier transform [10]. This closes the classical construction.

Let us now present how to adapt the previous construction when $\mathbb{C}[ISU(2)]$ is replaced by the quantum double $DSU(2)$. This idea is motivated by the fact that, in three dimensions, quantum gravity is argued to turn classical isometry group algebras into quantum groups [12]. In particular, $\mathbb{C}[ISU(2)]$ is deformed into

$DSU(2)$ when quantizing three dimensional Euclidean gravity without cosmological constant, the quantum deformation parameter being the Planck length ℓ_P. Any element of $DSU(2)$ can be written as $(f \otimes u)$ where $f \in C(SU(2))$ (one can extend the definition to distributions) are interpreted as "deformed" translational elements and $u \in \mathbb{C}[SU(2)]$; when $u \in SU(2)$ it is interpreted as a rotational element.

Following the classical construction, we claim that the space of $SU(2)$-invariant linear forms on $DSU(2)$ is a representation of C. This space can be identified with the convolution algebra $C(SU(2))^*$ of distributions on $SU(2)$: its algebra structure has been obtained, as in the classical case, from the Hopf algebra duality procedure. The duality bracket between a distribution $\phi \in C(SU(2))^*$ and a function f in $C(SU(2))$ will be denoted $\langle f, \phi \rangle$ in the sequel. When ϕ is a function, the duality bracket can be given in terms of the (normalized) $SU(2)$ Haar-measure $d\mu$ as follows:

$$\langle f, \phi \rangle \ = \ \int d\mu(u) \, \overline{f(u)} \, \phi(u) \, . \tag{2}$$

Of course, C is a non-commutative algebra which can be interpreted as a deformation of the classical algebra $C(\mathbb{R}^3)^*$ of distributions on the momenta space $\mathbb{R}^3$. It is, in fact, well-known that the momenta space of a particle becomes curved in three dimensional Euclidean (quantum) gravity and the standard momenta space is replaced by the Lie group $SU(2)$. By construction, $C(SU(2))^*$ provides a representation space of $DSU(2)$ which can be interpreted, in that way, as a symmetry algebra of $C(SU(2))^*$ whose action will be denoted $\rhd$. More precisely, translations elements are functions on $SU(2)$ and acts by multiplication on $C(SU(2))^*$ whereas rotational elements are $SU(2)$ elements and act by the adjoint action:

$$\forall \, \phi \in C(SU(2))^* \quad f \rhd \phi = f\phi \quad \text{and} \quad u \rhd \phi = \mathrm{Ad}_u \phi \, . \tag{3}$$

The adjoint action is defined by the relation $\langle f, \mathrm{Ad}_u \phi \rangle = \langle \mathrm{Ad}_{u^{-1}} f, \phi \rangle$ with $\mathrm{Ad}_u f(x) = f(u^{-1}xu)$ for any u, x in $SU(2)$.

2.2   The fuzzy space formulation

Thus, we have a clear definition of the deformed space of momenta. To get the quantum analoguous of the space $C(\mathbb{E}^3)$ itself, we need to introduce a Fourier transform on $C(SU(2))^*$. This is done making use of harmonic analysis on the group $SU(2)$: the Fourier transform of a given $SU(2)$-distribution is the decomposition of that distribution into (the whole set or a subset of) unitary irreducible representations (UIR) of $SU(2)$. These UIR are labelled by a spin j, they are finite dimensional of dimension $d_j = 2j + 1$. The Fourier transform is an algebra morphism which is explicitely defined by:

$$\mathcal{F}: \ C(SU(2))^* \ \longrightarrow \ \mathrm{Mat}(\mathbb{C}) \equiv \bigoplus_{j=0}^{\infty} \mathrm{Mat}_{d_j}(\mathbb{C}) \tag{4}$$

$$\phi \ \longmapsto \ \widehat{\Phi} \equiv \mathcal{F}[\phi] = \oplus_j \mathcal{F}[\phi]^j \ = \ \oplus_j (\phi \circ D^j)(e) \tag{5}$$

where $\mathrm{Mat}_d(\mathbb{C})$ is the set of d dimensional complex matrices, D^j_{mn} are the Wiegner functions and $\circ$ is the convolution product. When ϕ is a function, its Fourier matrix components are obtained performing the following integral

$$\mathcal{F}[\phi]^j_{mn} \equiv \int d\mu(u)\,\phi(u)\,D^j_{mn}(u^{-1})\,. \tag{6}$$

The inverse map $\mathcal{F}^{-1} : \mathrm{Mat}(\mathbb{C}) \to C(SU(2))^*$ associates to any family of matrices $\widehat{\Phi} = \oplus_j \widehat{\Phi}^j$ a distribution according to the formula:

$$\langle f, \mathcal{F}^{-1}[\widehat{\Phi}]\rangle \;=\; \sum_j d_j \int d\mu(u)\overline{f(u)}\,\mathrm{tr}(\widehat{\Phi}^j\,D^j(u)) \;\equiv\; \int d\mu(u)\overline{f(u)}\,\mathrm{Tr}(\widehat{\Phi}D(u)) \tag{7}$$

for any function $f \in C(SU(2))$. We have introduced the notations $D = \oplus_j D^j$ and $\mathrm{Tr}\widehat{\Phi} = \sum_j d_j \mathrm{tr}(\widehat{\Phi}^j)$. Therefore, it is natural to interpret the algebra $\mathrm{Mat}(\mathbb{C})$ as a deformation of the classical algebra $C(\mathbb{E}^3)$ and then three dimensional Euclidean quantum geometry is fundamentally non-commutative and fuzzy.

2.3 Relation to $C(\mathbb{E}^3)$: the non-commutative algebra $C_{\ell_P}(\mathbb{E}^3)$

It is not completely trivial to view how the algebra of matrices $\mathrm{Mat}(\mathbb{C})$ is a deformation of the classical algebra of functions on $\mathbb{E}^3$.

To make it more concrete, it is necessary to construct a precise link between $C(SU(2))^*$ and $C(\mathbb{R}^3)^*$ for the former space is supposed to be a deformation of the later. First, we remark that it is not possible to find a vector space isomorphism between them because $SU(2)$ and $\mathbb{R}^3$ are not homeomorphic: in more physical words, there is no way to establish a one to one mapping between distributions on $SU(2)$ and distributions on $\mathbb{R}^3$ for $SU(2)$ and $\mathbb{R}^3$ have different topologies. Making an explicit link between these two spaces is in fact quite involved and one construction has been proposed in a companion paper [10]. The aim of this Section is to recall only the main lines of that construction; more details can be found in [10]. For pedagogical reasons, we also restrict the space $C(SU(2))^*$ to its subspace $C(SU(2))$ and then we are going to present the link between $C(SU(2))$ and $C(\mathbb{R}^3)$.

1. First, we need to introduce a parametrization of $SU(2)$: $SU(2)$ is identified with $S^3 = \{(\vec{y}, y_4) \in \mathbb{R}^4 | y^2 + y_4^2 = 1\}$ and any $u \in SU(2)$ is given by

$$u(\vec{y}, y_4) \;=\; y_4 - i\vec{y}\cdot\vec{\sigma} \tag{8}$$

in the fundamental representation in terms of the Pauli matrices σ_i. For later convenience, we cut $SU(2)$ in two parts: the north hemisphere U_+ ($y_4 > 0$) and the south hemisphere U_- ($y_4 < 0$).

2. Then, we construct bijections between the spaces $U_\pm$ and the open ball of $\mathbb{R}^3$ $B_{\ell_P} = \{\vec{p} \in \mathbb{R}^3 | p < \ell_P^{-1}\}$: to each element $u \in U_\pm$ we associate a vector $\vec{P}(u) = \ell_P^{-1}\vec{y}$. This bijections implicitly identify $\vec{P}(u)$ with the physical momenta of

the theory. Note that this is a matter of choice: on could have chosen another expression for $\vec{P}(u)$ and there is no physical arguments to distinguish one from the other. We made what seems to be, for different reasons, the more natural and the more convenient choice.

3. As a consequence, any function $\phi \in C(SU(2))$ is associated to a pair of functions $\phi_\pm \in C(U_\pm)$, themselves being associated, using the previous bijections, to a pair of functions $\psi_\pm \in C_{B_{\ell_P}}(\mathbb{R}^3)$ which are functions on $\mathbb{R}^3$ with support on the ball B_{ℓ_P}. In that way, we construct two mappings $\mathfrak{a}_\pm : C(U_\pm) \to C_{B_{\ell_P}}(\mathbb{R}^3)$ such that $\mathfrak{a}_\pm(\phi_\pm) = \psi_\pm$ are explicitely given by:

$$\psi_\pm(\vec{p}) = \int d\mu(u)\delta^3(\vec{p} - \vec{P}(u))\phi_\pm(u) = \frac{v_{\ell_P}}{\sqrt{1 - \ell_P^2 p^2}}\phi(u(\ell_P\vec{p}, \pm\sqrt{1 - \ell_P^2 p^2})) \quad (9)$$

where $v_{\ell_P} = \ell_P^3/2\pi^2$. Then we have established a vector space isomorphism $\mathfrak{a} = \mathfrak{a}_+ \oplus \mathfrak{a}_-$ between $C(SU(2))$ and $C_{B_{\ell_P}}(\mathbb{R}^3) \oplus C_{B_{\ell_P}}(\mathbb{R}^3)$. We need two functions on $\mathbb{R}^3$ to characterize one function of $C(SU(2))$. The mapping $\mathfrak{a}_\pm$ satisfies the important following property: the action of the Poincaré group $ISU(2) \subset DSU(2)$ on $C_{B_{\ell_P}}(\mathbb{R}^3)$ induced by the mappings $a_\pm$ is the standard covariant one, namely

$$\xi \triangleright \mathfrak{a}_\pm(\phi_\pm) = \mathfrak{a}_\pm(\xi \triangleright \phi_\pm) \quad \forall \xi \in ISU(2) \subset DSU(2) . \quad (10)$$

In the r.h.s. (resp. l.h.s.), $\triangleright$ denotes the action of $\xi \in ISU(2)$ (resp. ξ viewed as an element of $DSU(2)$) on $C(\mathbb{R}^3)$ (resp. $C(SU(2))$). This was in fact the defining property of the mappings $\mathfrak{a}_\pm$.

Now, we have a precise relation between $C(SU(2))$ and $C(\mathbb{R}^3)$. Using the standard Fourier transform $\mathfrak{F} : C(\mathbb{R}^3)^* \to C(\mathbb{E}^3)$ restricted to $C_{B_{\ell_P}}(\mathbb{R}^3)$, one obtains the following mapping:

$$\mathfrak{m} \equiv \mathfrak{F} \circ \mathfrak{a} : C(SU(2)) \longrightarrow C_{\ell_P}(\mathbb{E}^3) \quad (11)$$

where $C_{\ell_P}(\mathbb{E}^3)$ is defined as the image of $C(SU(2))$ by $\mathfrak{m}$. It will be convenient to introduce the obvious notation $\mathfrak{m} = \mathfrak{m}_+ \oplus \mathfrak{m}_-$. We have the vector spaces isomorphism $C_{\ell_P}(\mathbb{E}^3) \simeq \widetilde{C}_{B_{\ell_P}}(\mathbb{R}^3) \oplus \widetilde{C}_{B_{\ell_P}}(\mathbb{R}^3)$ where $\widetilde{C}_{B_{\ell_P}}(\mathbb{R}^3)$ is the subspace of functions on $\mathbb{E}^3$ whose spectra is strictly contained in the open ball B_{ℓ_P} of radius ℓ_P^{-1}. Elements of $C_{\ell_P}(\mathbb{E}^3)$ are denoted $\Phi_+ \oplus \Phi_-$ where $\Phi_\pm(x) \in \widetilde{C}_{B_{\ell_P}}(\mathbb{R}^3)$. The explicit relation between $C(SU(2))$ and $C_{\ell_P}(\mathbb{E}^3)$ is

$$\Phi_\pm(x) \equiv \mathfrak{m}_\pm(\phi_\pm)(x) = \int d\mu(u)\phi_\pm(u) \exp(iP(u) \cdot x) . \quad (12)$$

This transform is clearly invertible. Note that, in [10], $C_{\ell_P}(\mathbb{E}^3)$ is the image of the whole algebra of distributions $C(SU(2))^*$: in that case, $C_{\ell_P}(\mathbb{E}^3)$ is the direct sum of three sub-spaces of $C(\mathbb{E}^3)$, two of them being isomorphic to the space of

distributions on $\mathbb{E}^3$ with support on B_{ℓ_P}, the last one being isomorphic to the space of distributions on $\mathbb{E}^3$ with support on ∂B_{ℓ_P}.

It remains to establish the link between $C_{\ell_P}(\mathbb{E}^3)$ and the space of matrices $\text{Mat}(\mathbb{C})$. To do so, we make use of the mapping $\mathcal{F}$ between $C(SU(2))$ and $\text{Mat}(\mathbb{C})$ and the mapping $\mathfrak{m}$ between the same $C(SU(2))$ and $C_{\ell_P}(\mathbb{E}^3)$. If we denote by $\widehat{\Phi}_\pm$ the images of $\phi_\pm$ by $\mathcal{F}$ then we have:

$$\Phi_\pm(x) \;=\; \text{Tr}(K_\pm^\dagger(x)\widehat{\Phi}_\pm) \tag{13}$$

where $K_\pm$ can be interpreted as the components of the element $K \equiv K_+ \oplus K_- \in \text{Mat}(\mathbb{C}) \otimes C_{\ell_P}(\mathbb{E}^3)$ defined by the integral:

$$K_\pm(x) \;\equiv\; \int_{U_\pm} d\mu(u)\, D(u)\, \exp(-iP(u)\cdot x) \,. \tag{14}$$

The relation (13) is invertible. One can interpret the functions $\Phi_\pm(x)$ as a kind of continuation to the whole Euclidean space of the discrete functions $\widehat{\Phi}_{\pm mn}^j$ which are a priori defined only on a infinite but numerable set of points. Given $x \in \mathbb{E}^3$, each matrix element $\widehat{\Phi}_{\pm mn}^j$ contributes to the definition of $\Phi_\pm(x)$ with a complex weight $\overline{K_{\pm nm}^j(x)}$.

For the moment, we have only described the vector space structure of $C_{\ell_P}(\mathbb{E}^3)$. However, this space inherits a non-commutative algebra structure when we ask the mapping $\mathfrak{m}$ to be an algebra morphism. The product between two elements Φ_1 and Φ_2 in $C_{\ell_P}(\mathbb{E}^3)$ is denoted $\Phi_1 \star \Phi_2$ and is induced from the convolution product $\circ$ on $C(SU(2))$ as follows:

$$\Phi_1 \star \Phi_2 \;=\; \mathfrak{m}(\mathfrak{m}^{-1}(\Phi_1) \circ \mathfrak{m}^{-1}(\Phi_2)) \,. \tag{15}$$

The $\star$-product is a deformation of the classical pointwise product. A very similar $\star$-product has been introduced in [9] in the context of Spin-Foam models; the main difference being that their algebra consists in only one copy of $C_{B_{\ell_P}}(\mathbb{E}^3)$ and then appears to be not clearly related to $C(SU(2))$.

In order to make the $\star$-product more intuitive, it might be useful to consider some examples of products of functions. The more interesting functions to consider first are surely the plane waves. Unfortunately, plane waves are not elements of $C(SU(2))$ but are pure distributions and then, their studies goes beyond what we recalled in this paper. Nevertheless, we will see that it is possible to extend the previously presented results to the case of the plane waves with some assumptions. Plane waves are defined as eigenstates of the generators P_a and then, as we have already underlined, a plane wave is represented by the distribution δ_u with eigenvalue $P_a(u)$ which is interpreted as the momentum of the plane wave. Plane waves are clearly degenerated as $P_a(u)$ is not invertible in $SU(2)$: this result illustrates the fact that we need two functions $\Phi_+ \oplus \Phi_- \in C_{\ell_P}(\mathbb{E}^3)$ to characterize one function

$\phi \in C(SU(2))$. The representations of the plane wave in the matrix space $\mathrm{Mat}(\mathbb{C})$ and in the continuous space $C_{\ell_P}(\mathbb{E}^3)$ are respectively given by:

$$\mathcal{F}(\delta_u)^j = D^j(u)^{-1} \quad \text{and} \quad \mathfrak{m}(\delta_u)(x) \equiv w_u(x) \tag{16}$$

where $w_u(x) = \exp(iP_a(u)x^a)\oplus 0$ if $u \in U_+$ and $w_u(x) = 0\oplus\exp(iP_a(u)x^a)$ if $u \in U_-$. The framework we have described do not include the case $u \in \partial U_+ = \partial U_-$ which is nonetheless completely considered in [10]. The $\star$-product between two plane waves reads:

$$w_u \star w_v = w_{uv} \tag{17}$$

if u, v and uv belongs to U_+ or U_-. This product can be trivially extended to the cases where the group elements belong to the boundary $\partial U_+ = \partial U_-$. As a result, one interprets $P_a(u) \boxplus P_a(v) \equiv P_a(uv)$ as the deformed addition rule of momenta in the non-commutative space.

Other interesting examples to consider are the coordinate functions. They are easily defined using the plane waves and their definition in the $C(SU(2))^*$ and $\mathrm{Mat}(\mathbb{C})$ representations are:

$$\chi_a = 2i\ell_P\xi_a\delta_e \in C(SU(2))^* \qquad \widehat{x}_a = 2\ell_P D(J_a) \in \mathrm{Mat}(\mathbb{C}) \tag{18}$$

where ξ_a is the $SU(2)$ left-invariant vector field and J_a the generators of the $\mathfrak{su}(2)$ Lie algebra satisfying $[J_a, J_b] = 2i\epsilon_{ab}{}^c J_c$. In the $C_{\ell_P}(\mathbb{E}^3)$ representation, the coordinates are given by $X_a \equiv (x_a \oplus 0)$; only the first component is non-trivial. It becomes straightforward to show that the coordinates satisfy the relation

$$[X_a, X_b]_\star \equiv X_a \star X_b - X_b \star X_a = i\ell_P\epsilon_{ab}{}^c X_c \tag{19}$$

and therefore do not commute as expected.

We end this Section by a quick summary of the different representations of the (suitable sub-algebra of the) algebra C: giving a function $\phi \in C(SU(2))$ is equivalent to give either a couple of functions $\psi_\pm = \mathfrak{a}_\pm(\phi)$ which belong to $C(\mathbb{R}^3)$; or a couple of functions $\Phi_\pm = \mathfrak{m}_\pm(\phi) \in C(\mathbb{E}^3)$; or a matrix $\widehat{\Phi} = \mathcal{F}(\phi) \in \mathrm{Mat}(\mathbb{C})$ or finally a couple of matrices $\widehat{\Phi}_\pm = \mathcal{F}(\phi_\pm) \in \mathrm{Mat}(\mathbb{C})$.

2.4 An integral on the non-commutative algebra

An important property is that the non-commutative space admits an invariant measure $h : C \to \mathbb{C}$. To be more precise, h is well defined on the restriction of $C \simeq C(SU(2))^*$ to $C(SU(2))$. The invariance is defined with respect to the symmetry action of the Hopf algebra $DSU(2)$. Let us give the expression of this invariant measure in the different formulations of the non-commutative space:

$$h(\phi) = \phi(e) = \mathrm{Tr}(\widehat{\Phi}) = \int \frac{d^3x}{(2\pi)^3 v_{\ell_P}} \Phi_+(x) \tag{20}$$

where $\phi \in C(SU(2))$, $\widehat{\Phi} = \mathcal{F}[\phi]$ and $\Phi_+(x) = \mathfrak{m}_+[\phi](x)$. Note that $\int d^3x$ is the standard Lebesgue measure on the classical manifold $\mathbb{E}^3$. This measure can be extended to distributions which are well-defined at the origin e in the sense that they behave like regular functions at the vicinity of e.

Sometimes, such a measure is called a trace. It allows to define a norm on the algebra C from the hermitian bilinear form

$$\langle \phi_1, \phi_2 \rangle \equiv h(\phi_1^\flat \phi_2) = \int d\mu(u) \, \overline{\phi_1(u)} \phi_2(u) \tag{21}$$

where $\phi^\flat(u) = \overline{\phi(u^{-1})}$. As we will see below, such a trace is necessary to define an action for a field living on the non-commutative space.

2.5 Derivative operators

Derivative operators ∂_ξ can be deduced from the action of infinitesimal translations: given a vector $\xi \in \mathbb{E}^3$, we have $\partial_\xi = \xi^a \partial_a$ where $\partial_a = iP_a$ is the translation operator we have introduced in the previous section. When acting on the $C(SU(2))$ representation, ∂_ξ is the multiplication by the function $i\xi^a P_a$; it is the standard derivative when acting on the continous $C_{\ell_P}(\mathbb{E}^3)$ representation (using the mapping $\mathfrak{m}$); finally it is a finite difference operator when acting on the fuzzy space representation $\mathrm{Mat}(\mathbb{C})$ (using the Fourier transform $\mathcal{F}$). After some calculations, one shows that its expression in the matrix representation is then given by the following:

$$
\begin{aligned}
(\partial_a \widehat{\Phi})^j &= \mathrm{tr}_{j-1/2}\left[(\widehat{\Phi}^{j-1/2} \otimes \mathbb{I}) \cdot C_a(j - 1/2, j) \right] \\
&\quad + \mathrm{tr}_{j+1/2}\left[(\widehat{\Phi}^{j+1/2} \otimes \mathbb{I}) \cdot C_a(j + 1/2, j) \right]
\end{aligned}
$$

where we have introduced the operator $C_a(j,k) \in \mathrm{Mat}_{d_j}(\mathbb{C}) \otimes \mathrm{Mat}_{d_k}(\mathbb{C})$:

$$C_a(j,k) \equiv -\frac{1}{\ell_P} \int d\mu(u) \, \mathrm{tr}[D^{1/2}(J_a u)] \, \overline{D^j(u)} \otimes D^k(u) . \tag{22}$$

The notation tr_j means that we perform a trace in the space of dimension d_j. The matrix coefficients of the operator $D_a(j,k)$ can be explicitly computed in terms of $SU(2)$ Clebsh-Gordan coefficients and we finally get the following operator

$$
\begin{aligned}
(\partial_a \widehat{\Phi})^j_{st} &= -\frac{1}{\ell_P d_j} D^{1/2}_{pq}(J_a) \; (\sqrt{(j + 1 + 2qs)(j + 1 + 2tp)} \, \widehat{\Phi}^{j+1/2}_{q+s\,p+t} \\
&\qquad + (-1)^{q-p} \sqrt{(j - 2qs)(j - 2pt)} \, \widehat{\Phi}^{j-1/2}_{q+s\,p+t}). \tag{23}
\end{aligned}
$$

Details of the calculation can be found in the appendix of the companion paper [10]. The interpretation of the formula (23) is clear. Note however an important point: the formula (23) defines a second order operator in the sense that it involves $\widehat{\Phi}^{j-1/2}$ and $\widehat{\Phi}^{j+1/2}$ that are not nearest matrices but second nearest matrices.

The derivative operator is obviously necessary to define a dynamics in the non-commutative fuzzy space. The ambiguity in the definition of P_a implies immediately

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an ambiguity in the dynamics. For instance, the fact that $C_a(j,k)$ relates matrices $\widehat{\Phi}^j$ with $\widehat{\Phi}^{j\pm1/2}$ only is a consequence of the choice of P_a which is in fact a function whose non-vanishing Fourier modes are the matrix elements of a dimension 2 matrix: indeed, $P_a(u) = \ell_P^{-1}\mathrm{tr}_{1/2}(J_a u)$. Another choice would lead to a different dynamics and then there is ambiguity. Such ambiguities exists as well in full LQG.

2.6 Free field: solutions and properties

Now, we have all the ingredients to study dynamics on the quantum space. Due to the fuzzyness of space, equations of motion will be discrete and therefore, there is in general no equivalence between Lagrangian and Hamiltonian dynamics. Here, we choose to work in the Euler-Lagrange point of view, i.e. the dynamics is governed by an action of the type:

$$S_\star[\Phi, J] \;=\; \frac{1}{2}\int \frac{d^3x}{(2\pi)^3 v_{\ell_P}} \;\left(\partial_\mu\Phi \star \partial_\mu\Phi \;+\; V(\Phi, J)\right)_+(x) \tag{24}$$

where V is the potential that depends on the field Φ and eventually on some exterior fields J. The action has been written in the $C_{\ell_P}(\mathbb{E}^3)$ formulation to mimic easily the classical situation. However, one has to be aware that Φ comes from an element $\phi \in C(SU(2))$ in the sense that $\Phi(x) = \mathfrak{m}(\phi)(x)$ and therefore cannot be any classical function on $\mathbb{E}^3$; in particular, it has a bounded spectrum. The integral we use to define the action is the measure introduced in previous sections (20).

Finding the equations of motions reduces obviously in extremizing the previous action, but with the constraint that Φ belongs to $C_{\ell_P}(\mathbb{E}^3)$: in particular, Φ (as well as the exterior field) admits two independent components $\Phi_\pm$ which are classical functions on $\mathbb{E}^3$ whose spectra are bounded. The action (24) couples generically these two components. Even when one of the two fields vanishes, for instance $\Phi_- = 0$, it happens in general that the extrema of the functional $S[\Phi]$ differ from the ones that we obtain for a classical field Φ whose action would be formally the same functional but defined with the pointwise product instead of the $\star$ product. This makes the classical solutions in the deformed and undeformed cases different in general. Let us precise this point. When the field is free in the sense that V is quadratic (with a mass term), deformed solutions are the same as classical ones. However, solutions are very different when the dynamics is non-linear and the differences are physically important. It is the purpose of this paper to illustrate this fact in some simple examples.

First, let us consider the case of a free field: we assume that $V(\Phi) = \mu^2\Phi \star \Phi$ where μ is a positive parameter. Equations of motion are obtained by extremizing the action with the constraints that $\Phi = \mathfrak{m}(\phi)$, ϕ being in $C(SU(2))$. These equations are best written in the fuzzy space formulation and one gets as expected the following set of finite difference equations:

$$\Delta\widehat{\Phi}^j \;+\; \mu^2\widehat{\Phi}^j \;=\; 0 \quad \text{for all spin } j. \tag{25}$$

Due to the quite complicated expression of the derivative operator, it appears more convenient to solve this set of equations in the $C(SU(2))$ representation. Indeed, these equations are equivalent to the fact that $\phi = \mathcal{F}^{-1}[\Phi]$ has a support in the conjugacy classes $\theta \in [0, 2\pi[$ such that $\sin^2(\theta/2) = \ell_P^2 \mu^2$. Thus, a solution exists only if $\mu \leq \ell_P^{-1}$, in which case we write $\mu = \ell_P^{-1} \sin(m/2)$ with $0 < m < \pi$. Then the solutions of the previous system are given by $\widehat{\Phi} = \widehat{\Phi}_+ + \widehat{\Phi}_-$ with:

$$\widehat{\Phi}_\pm^j = \int d\mu(u) \, \mathbb{I}_m^\pm(u) \left(\alpha(u) D^j(u) + \beta(u) D^j(u)^\dagger \right) \tag{26}$$

where α and β are $SU(2)$ complex valued functions; the notation $\mathbb{I}_m^\pm$ holds for the caracteristic functions on the conjugacy class $\theta = m$ (for the $+$ sign) and $\theta = 2\pi - m$ (for the $-$ sign). These functions are normalized to one according to the relation $\int d\mu(u) \mathbb{I}_m^\pm(u) = 1$. If the fields $\Phi_\pm(x)$ are supposed to be real, the matrices $\widehat{\Phi}^j$ are hermitian, and then α and β are complex conjugate functions. As a result, we obtain the general solution for the non-commutative free field written in the fuzzy space representation.

Using the mapping $\mathfrak{m}$, one can reformulate this solution in terms of functions on $\mathbb{E}^3$. The components of Φ are given by:

$$\Phi_\pm(x) = \frac{\ell_P^2}{16\pi} \frac{\sin^2 \frac{m}{2}}{\cos \frac{m}{2}} \int_{B_{\ell_P}} d^3p \, \delta(p - \mu) \left(\alpha_\pm(p) e^{ip \cdot x} + \beta_\pm(p) e^{-ip \cdot x} \right) \tag{27}$$

where B_{ℓ_P} is the Planck ball, $\alpha_\pm(p) = \alpha(u(p))$ where $u(p)$ is the inverse of $p(u)$ when u is restricted to the sets $U_\pm$; a similar definition holds for $\beta_\pm$. We recover the usual solution for classical free scalar fields with the fact that the mass has an upper limit given by ℓ_P^{-1}. Therefore, the Planck mass appears to be a natural UV cut-off. This result can a priori be extended to any free (quadratic) field theory, like Dirac or Maxwell theory for instance. We hope to study these important examples in future works.

3   Particles evolving in the fuzzy space

Important discrepencies between classical and fuzzy dynamics appear when one considers non-linear interactions. In the case we study the dynamics of a sole field ϕ, one has to introduce self-interactions. However, even in the standard classical commutative space $\mathbb{E}^3$, classical solutions of self-interacting field cannot be written in a closed form in general; and then one cannot expect to find explicit solutions for the self-interacting field evolving in the fuzzy background. Face with such technical difficulties (that we postpone for future investigations), we will consider simpler models. We will perform symmetry reductions in order that the field ϕ depends only on one coordinate out of the three. We will interpret this model as describing one particle evolving in (Euclidean) fuzzy space-time.

3.1 Reduction to one dimension

Let us define the algebra C^{1D} of symmetry reduced fields and its different representations: the group algebra, the matrix and the continuous formulations. Using a trivial analogy with the classical case, C^{1D} is defined as the kernel of the operators P_1 and P_2 in the convolution algebra $C(SU(2))^*$ where P_a are the momentum coordinates:

$$C^{1D} \simeq \{\phi \in C(SU(2))^* \mid \phi = \varphi(P_1)\delta(P_2)\delta(P_3)\} . \tag{28}$$

As a result, C^{1D} can be identified to the set $C(U(1))^*$ of $U(1)$ distributions. This set inherits an algebra structure from the product on the full algebra C: it is the $U(1)$ convolution product. Note that, the algebra becomes commutative but, as we will see in the sequel, the product is still non-trivial and exhibits interesting properties compared to the classical one. In the sequel, we identify ϕ of C^{1D} with the $U(1)$ distributions φ (28) and we choose a parametrization such that φ is a function of $\theta \in [0, 2\pi]$. The algebra C^{1D} admits two other formulations: the matrix one obtained from the induced Fourier transform and the continuous one obtained from the induced map $\mathfrak{m}$.

Let us first consider the matrix representation. A priori, any element $\varphi \in C^{1D}$ admits as a Fourier transform an infinite set of matrices. This set is in fact highly degenerate due to the symmetry reduction and reduces to only one infinite dimensional diagonal matrix $\widehat{\Phi} \in \mathrm{Diag}_\infty(\mathbb{C})$. The relation between the diagonal matrix elements $\widehat{\Phi}_a^a$ and the associated distribution φ is given by:

$$\mathcal{F}^{1D} : C(U(1))^* \longrightarrow \mathrm{Diag}_\infty(\mathbb{C}) , \quad \varphi \longmapsto \widehat{\Phi} \text{ with } \widehat{\Phi}_a^a \equiv \varphi_a = \langle \varphi, e^{ia\theta} \rangle \tag{29}$$

where $\langle , \rangle$ is the duality bracket between $U(1)$ distributions and $U(1)$ functions. This identity reduces to the more concrete following relation when φ is supposed to be a function:

$$\varphi_a = \frac{1}{2\pi} \int_0^{2\pi} d\theta \, \varphi(\theta) e^{ia\theta} . \tag{30}$$

Thus, the non-commutative Fourier transform reduces to the simple Fourier modes decomposition of a periodic one-dimensional function. Indeed, we have $\mathrm{Diag}_\infty(\mathbb{C}) \simeq \mathbb{Z} \otimes \mathbb{C}$ which is the Fourier space of $U(1)$ distributions. The algebra structure of $\mathrm{Diag}_\infty(\mathbb{C})$ is induced from the convolution product $\circ$ and is simply given by the commutative discrete pointwise product:

$$\forall \, \varphi, \varphi' \in C(U(1))^* \quad (\varphi \circ \varphi')_a = \varphi_a \, \varphi_a' . \tag{31}$$

Let us now construct the mapping between the convolution algebra $C(U(1))$ and the algebra $C_{\ell_P}(\mathbb{E}^1)$ which has to be understood for the moment as the one-dimensional analoguous of $C_{\ell_P}(\mathbb{E}^3)$. We proceed in the same way as in the full theory:

1. first, we cut $U(1) \equiv [0, 2\pi]$ in two parts, $U_+ \equiv]-\frac{\pi}{2}, \frac{\pi}{2}[$ and $U_- \equiv]\frac{\pi}{2}, \frac{3\pi}{2}[$ where the symbol $\equiv$ means equal modulo 2π;

2. then, we construct two bijections between $U_\pm$ and $B_{\ell_P}^{1D} \equiv]-\ell_P^{-1}; \ell_P^{-1}[$ by assigning to each $\theta \in U_\pm$ a momentum $P(\theta) = \ell_P^{-1} \sin \theta$;

3. the third step consists in associating to any function $\varphi \in C(U(1))$ a pair of functions $\varphi_\pm \in C(U_\pm)$, and a pair of functions $\psi_\pm \in C(\mathbb{R})$ induced by the previous bijections as follows

$$
\begin{aligned}
\mathfrak{a}_\pm^{1D}(\phi_\pm)(p) \equiv \psi_\pm(p) &= \int \frac{d\theta}{2\pi} \delta(p - \ell_P^{-1} \sin \theta)\, \varphi_\pm(\theta) \\
&= \frac{\ell_P}{4\pi} \frac{1}{\sqrt{1 - \ell_P^2 p^2}} \varphi_\pm(\theta(p))
\end{aligned}
\tag{32}
$$

where $\theta(p)$ is the inverse of $p(\theta) = \ell_P^{-1} \sin \theta$ in each open $U_\pm$;

4. finally, we make use of the standard one dimensional Fourier transform $\mathfrak{F}^{1D}$ to construct the mapping $\mathfrak{m}^{1D} = \mathfrak{m}_+^{1D} \oplus \mathfrak{m}_-^{1D} : C(U(1)) \to C_{\ell_P}(\mathbb{E}^1)$ where the components $\mathfrak{m}_\pm^{1D} = \mathfrak{F}^{1D} \circ \mathfrak{a}_\pm^{1D}$ are given by:

$$
\mathfrak{m}_\pm^{1D}(\varphi_\pm)(t) \equiv \Phi_\pm(t) = \int_0^{2\pi} \frac{d\theta}{2\pi} \varphi_\pm(\theta) \exp(iP(\theta)t) .
\tag{33}
$$

The space $C_{\ell_P}(\mathbb{E}^1)$ is the image of $C(U(1))$ by $\mathfrak{m}$ and therefore is defined by $\widetilde{C}(U_+) \oplus \widetilde{C}(U_-)$ where $\widetilde{C}(U_\pm)$ are the image by $\mathfrak{F}^{1D}$ of $C(U_\pm)$. As in the full theory, this construction can be extended to the algebra $C(U(1))^*$ of distributions.

It remains to construct the link between the discrete and the continuous representations of C^{1D}. To do so, we compose the Fourier transform with the map $\mathfrak{m}^{1D}$, and we obtain the reduced version of the formula (13) linking $\Phi_\pm(t)$ with φ_a:

$$
\Phi_\pm(t) = \sum_a \varphi_a K_\pm^a(t)
\tag{34}
$$

where the functions $K_\pm^a(t)$ are defined by the integrals

$$
K_\pm^a(t) \equiv \int_{U_\pm} \frac{d\theta}{2\pi} e^{-ia\theta + iP(\theta)t} = (\pm 1)^a \int_0^{\frac{\pi}{2}} \frac{d\theta}{\pi} \cos\left(a\theta \mp \frac{t}{\ell_P} \sin \theta\right) .
\tag{35}
$$

As in the general case, the relation (34) is invertible. The integral defining $K_\pm$ is a simplified version of the general formula (14) and one can viewed these functions as the components of the element $K = K_+ \oplus K_- \in \mathrm{Diag}_\infty(\mathbb{C}) \otimes C_{\ell_P}(\mathbb{E}^1)$. Furthermore, $K^a = K_+^a \oplus K_-^a$ is the image by $\mathfrak{m}^{1D}$ of the (discrete) plane waves $\exp(-ia\theta)$. As a last remark, let us underline that K_+ and K_- are closely related by the property $K_-^a(-t) = (-1)^a K_+(t)$. This implies that the functions $\Phi_\pm$ are also closely related:

if we assume for instance that $\varphi_{2n+1} = 0$ for any $n \in \mathbb{Z}$ then $\Phi_-(-t) = \Phi_+(t)$; if we assume on the contrary that $\varphi_{2n} = 0$ for any $n \in \mathbb{Z}$ then $\Phi_-(-t) = -\Phi_+(t)$. Such a property will have physical consequences as we will see in the sequel.

Let us give some physical interpretation of the formula (34). One can view it as a way to extend φ_a, considered as a function on $\mathbb{Z}$, into the whole real line $\mathbb{R}$. In that sense, this formula is a link between the discrete quantum description of a field and a continous classical description. One sees that any microscopic time a contributes (positively or negatively) to the definition of a macroscopic time t with an amplitude precisely given by $K_{\pm}^a(t)$. At the classical limit $\ell_P \to 0$, $K_{\pm}^a(t)$ are maximal for values of the time $t = \pm \ell_P a$. In other words, the more the microscopic time $a\ell_P$ is close to the macroscopic time t, the more the amplitude $K_{\pm}^a(t)$ is important.

Concerning the reduced $\star$-product, it is completely determined by the algebra of the functions K^a viewed as elements of $C_{\ell_P}(\mathbb{E}^1)$ and a straightforward calculation leads to the following product between K^a type functions:

$$K^a \star K^b \equiv \mathfrak{m}^{1D}(\exp(-ia\theta) \circ \exp(-ib\theta)) = \delta^{ab} K_a . \qquad (36)$$

This result clearly illustrates the non-locality of the $\star$-product.

Before going to the dynamics, let us give the expression of the derivative operator ∂_t. As for the general case, ∂_t is a finite difference operator whose action on $\mathrm{Diag}_\infty(\mathbb{C})$ is given, as expected, by the following formula:

$$(\partial_t \varphi)_a = \frac{1}{2\ell_P} (\varphi_{a+1} - \varphi_{a-1}) . \qquad (37)$$

This expression is highly simplified compared to the more general one introduced in the previous section. However, we still have the property that ∂_t is in fact a second order operator for it relates $a + 1$ and $a - 1$. A important consequence would be that the dynamics (of the free field) will decouple the odd components φ_{2n} and the even components φ_{2n+1} of the discrete field. Then, we will have two independent dynamics which could be interpreted as two independent particles evolving in the fuzzy space. In particular, one could associated the continuous fields $\Phi(t)^{odd}$ and $\Phi(t)^{even}$ respectively associated to the families (φ_{2n}) and (φ_{2n+1}). It is clear that $\Phi(t)^{odd}$ and $\Phi(t)^{even}$ are completely independent one to the other and, using the basic properties of $K_\pm$, we find that the $\pm$ components of each field are related by:

$$\Phi_-^{odd}(-t) = \Phi_+^{odd}(t) \quad \text{and} \quad \Phi_-^{even}(-t) = -\Phi_+^{even}(t) . \qquad (38)$$

Thus, Φ_+ and Φ_- fundamentaly describe two "miror" particles.

3.2 Dynamics of a particle: linear vs. non linear

Now, we have all the ingredients to study the behavior of the one dimensional field φ. When written in the continuous representation, its dynamics is governed by an action of type (24) but one-dimensional only, with no external field J and the

potential is supposed to be monomial, i.e. of the form $V(\Phi) = \varepsilon/(\alpha+1)\Phi^{\star(\alpha+1)}$ with $\alpha + 1$ a non-null integer. The equations of motions are given by:

$$\Delta\Phi + \varepsilon\Phi^{\star\alpha} = 0 \tag{39}$$

where $\Delta = \partial_t^2$. Due to the form of the $\star$-product, these equations are generically ($\alpha > 1$) highly non-local and mix the two components $\Phi_\pm$ of the field. In fact, they are best written in the fuzzy space representation where they reduce to the following finite difference equation:

$$\frac{\varphi_{a+2} - 2\varphi_a + \varphi_{a-2}}{4\ell_P^2} = -\varepsilon\varphi_a^\alpha . \tag{40}$$

As it was previously emphasized, we note that these equations do not couple odd and even integers a. For simplicity purposes, we will consider only even spins, i.e. we assume that $\varphi_{2n+1} = 0$ for all integer n.

To warm up, let us start with a simple example: the case where the potential is those of a harmonic oscillator, i.e. $\alpha = 1$ and $\varepsilon = \Omega^2$. In that case, the system admits a simple exact solution given by:

$$\varphi_a = a_+\exp(i\omega_0\ell_P a) + a_-\exp(-i\omega_0\ell_P a) \tag{41}$$

where ω_0 satisfies the defining equation $\Omega^2\ell_P^2 = \sin^2(\omega_0\ell_P)$ together with (the restriction that) $\omega_0\ell_P \in [0, \pi/2]$ and then we have implicitely assumed that $\Omega\ell_P \leq 1$, which means that the period Ω^{-1} cannot be smaller than the Planck time. Otherwise, there is no oscillations and the amplitude of the motion decreases exponentially. Using the formula (34), one can extend this solution to the whole real line and one shows that the component $\Phi_\pm$ are explicitely given by:

$$\Phi_\pm(t) = a_+\exp(\pm iP(\omega_0\ell_P)t) + a_-\exp(\mp iP(\omega_0\ell_P)t) \tag{42}$$

where $P(\omega_0\ell_P) = \Omega$. It is interesting to note that the two components are simply related by $\Phi_+(t) = \Phi_-(-t)$: thus, Φ_+ and Φ_- have the same physical content; we will give an interpretation of that property in the sequel. As expected, the solution for $\Phi_\pm$ is the same as the standard classical one where the period of the oscillations is bounded. Nonetheless, the periods for the discrete field and the continuous field are different: one can interpret Ω as a renormalization of ω_0 due to gravitational effects.

This clearly shows that dynamics of a one-dimensional free field in the fuzzy space is very similar to those in a classical space. When the dynamics is non-linear, solutions are no longer the same and this section is devoted to illustrate this point.

For that purpose, we consider the dynamics (40) with $\alpha \geq 2$ and we look for perturbative solutions in the parameter ε. The corresponding classical solution Φ_c reads at the first order

$$\Phi_c(t) = vt - \varepsilon\frac{v^\alpha t^{\alpha+2}}{(\alpha+1)(\alpha+2)} + \mathcal{O}(\varepsilon^2) \tag{43}$$

where we assume for simplicity that $\Phi_c(0) = 0$ and $\Phi'_c(0) = v$.

The perturbative expansion of the fuzzy solution is obtained using the same techniques. We look for solutions of the type $\varphi_a = \lambda a + \varepsilon \eta_a$ where $a = 2k$ by asumption, λ is a real number and η must satisfy the following relation:

$$\eta_{2k} - \eta_{2k-2} = -\frac{\ell_P^2}{4} \lambda^\alpha \sum_{n=1}^{k-1} (2n)^\alpha = -\frac{\ell_P^2}{4}(2\lambda)^\alpha \left[\frac{(k-1)^{\alpha+1}}{\alpha+1} + \frac{(k-1)^\alpha}{2}\right.$$
$$+ \frac{\alpha(k-1)^{\alpha-1}}{12} - \frac{\alpha(\alpha-1)(\alpha-2)}{720}(k-1)^{\alpha-3}$$
$$+ \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{30240}(k-1)^{\alpha-5} + \cdots \right]$$

The solution is in general complicated. To be explicit, we will consider the case $\alpha = 2$. The formula simplifies a lot, and after some straightforward calculations, one shows that:

$$\eta_{2k} = -\frac{\ell_P^2 \lambda^2}{12} k^2 (k-1)(k+1) = -\frac{\ell_P^2 \lambda^2}{12}(k^4 - k^2). \tag{44}$$

In order to compute the $C_{\ell_P}(\mathbb{E}^1)$ representation of this solution, one uses the following relations for any integer n

$$S_\pm^{(n)}(t) \equiv \sum_{k=-\infty}^{+\infty} k^n K_\pm^{2k}(t) = \frac{1}{2(2i)^n} \frac{d^n}{d\theta^n} \exp(iP(\theta)t)|_{\frac{1\mp 1}{2}\pi}. \tag{45}$$

Applying this formula for $n = 1, 2$ and 4

$$S_\pm^{(1)}(t) = \pm\ell_P^{-1}t, \quad S_\pm^{(2)}(t) = 2\ell_P^{-2}t^2, \quad S_\pm^{(4)}(t) = 2(4\ell_P^{-4}t^4 + \ell_P^{-2}t^2) \tag{46}$$

one shows, after some simple calculations, that that Φ_+ and Φ_- are simply related by $\Phi_+(t) = \Phi_-(-t)$ and Φ_+ is given by:

$$\Phi_+(t) = 2\lambda\ell_P^{-1}t - \varepsilon\frac{\ell_P^2\lambda^2}{6}(2\ell_P^{-4}t^4 + \ell_P^{-2}t^2) + \mathcal{O}(\varepsilon^2). \tag{47}$$

To compare it with the classical solution Φ_c computed above (43), we impose the same initial conditions which leads to $\lambda = v\ell_P/2$ and then the solution reads:

$$\Phi_+(t) = vt - \varepsilon\frac{v^2t^4}{12} - \varepsilon\frac{\ell_P^2 v^2 t^2}{24} + \mathcal{O}(\varepsilon^2). \tag{48}$$

Let us interpret the solution. First, let us underline once again that Φ_+ and Φ_- are related by $\Phi_+(t) = \Phi_-(-t)$: thus, it seems that Φ_- corresponds to a particle evolving backwards compared to Φ_+. In that sense, the couple $\Phi_\pm$ behaves like a particle and a "miror" particle: the presence of the miror particle is due to quantum gravity effects. Second, we remark that the solution for Φ_+ differs from its classical counterpart at least order by order in the parameter ε. At the no-gravity limit $\ell_P \to 0$, Φ_+ tends to the classical solution (43). Therefore, we can interpret these discrepencies as an illustration of quantum gravity effects on the dynamics of a field.

3.3 Background independent dynamics

We finish this example with the question concerning the physical content of this solution. For the reasons we gave in the previous section, we concentrate only on the component Φ_+. Can one interpret $\Phi_+(t)$ as the position $q(t)$ of a particle evolving in the fuzzy space? If the answer is positive, it is quite confusing because the position should be discrete valued whereas Φ_+ takes value in the whole real line a priori. In fact, we would like to interpret $\Phi_+(t) = Q(t) \in \mathbb{R}$ as the extension in the whole real line of a discrete position $q(t) \in \mathbb{Z}$. More precisely, we suppose that the space where the particle evolves is one-dimensional and discrete, and then its motion should be caracterized by a $\mathbb{Z}$-valued function $q(t)$. If we restore the discreteness of the time variable, then the motion of the particle should be in fact caracterized by a set of ordered integers $\{q(2k\ell_P), k \in \mathbb{Z}\}$. To make this description more concrete, we make use of the identity satisfied by $S_+^{(1)}$ (46) which implies that:

$$Q(t) = \sum_{k=-\infty}^{+\infty} (2\ell_P k) \, K_+^{2k}(Q(t)) \ . \tag{49}$$

This identity makes clear that $Q(t)$ can be interpreted as a kind of continuation in the whole real line of a set of discrete positions and $K_+^{2k}(Q(t))$ gives the (positive or negative) weight of the discrete point $2\ell_P k$ in the evaluation of the continuous point $Q(t)$. Therefore, one can associate an amplitude $\mathcal{P}(k|\tau)$ to the particle when it is at the discrete position $Q = 2\ell_P k$ and at the discrete time $t = 2\ell_P \tau$ (in Planck units) in the fuzzy space. This amplitude is given by:

$$\mathcal{P}(k|\tau) = \frac{K_+^{2k}(Q(2\ell_P\tau))}{\sum_{j=-\infty}^{+\infty} K_+^{2j}(Q(2\ell_P\tau))} = K_+^{2k}(Q(2\ell_P\tau)) \tag{50}$$

because the normalisation factor equals one. These amplitudes cannot really be interpreted as statistical weight because they can be positive or negative. Nevertheless, they contain all the information of the dynamics of the particle in the sense that one can reconstruct the dynamic from these data. Therefore, we obtain a background independent description of the dynamics of the particle that can be a priori anywhere at any time: its position $2k\ell_P$ at a given time $2\tau\ell_P$ is caracterized by the amplitude previously defined. Furthermore, the amplitude is maximum around the classical trajectory, i.e. when $Q(2\ell_P\tau) = 2\ell_P k$, and gives back the classical trajectory at the classical limit defined by $k, \tau \to \infty$, $\ell_P \to 0$ with the products $k\ell_P$ and $\tau\ell_P$ respectively fixed to the values t (classical time) and Q (classical position).

We hope to generalize this interpretation for more general (relativistic) dynamics. described by a (continuous) vector $Q_\mu(s)$ which is a function of a parameter s (that can be the time component or something else). Indeed, there exists a relation generalizing (50) given by:

$$Q_\mu(s) = \ell_P \sum_j d_j \, \text{tr} \left(D^j(J_\mu) \, K_+^j(Q_\mu(s)) \right) \ . \tag{51}$$

The matrix-valued function K_+^j can be expressed in terms of special functions and its expression depends on the choice of the momenta functions P_a. Whatever the choice of P_a we make, these functions admits the same classical behavior.

Let us give an interpretation of this general formula. A fuzzy point is parametrized by its radius fixed by the representation j and its "angles" fixed by the magnetic numbers $i, j \in [-I, I]$. As the fuzzy radius R is a Casimir, it is possible to measure simultaneously the fuzzy radius and the fuzzy z component for instance. Then, we interpret the following fonction

$$\mathcal{P}(I, i|s) = d_j \, K^j(Q_\mu(s))_i^i \tag{52}$$

as the amplitude associated to a particle when it is on the sphere of radius $R = \ell_P \sqrt{I(I+1)}$ with $z = \ell_P i$. Thus, one would have a background independent description of the dynamics.

4 Discussion and perspectives

This article was mainly devoted to the study of the dynamics of a one-dimensional field in a given non-commutative geometry. This system is physically interpreted as a particle evolving in an Euclidean three-dimensional quantum geometry which is supposed to reproduce space at the Planck scale. In a first part, we have re-called the basic properties of this quantum background presenting in particular its different representations: the momentum space representation $C(SU(2))^*$, the fuzzy space representation $\mathrm{Mat}(\mathbb{C})$ and the continuous one $C_{\ell_P}(\mathbb{E}^3)$. We have constructed the basic ingredients to define a quantum field theory on such a space: an invariant integral and derivative operators. Then, we write the general action for a scalar field with the requirements that the action is local with respect to the non-commutative product and also "invariant" by the action of the deformed symmetry algebra $DSU(2)$. When the field is free, solutions are similar to classical ones (i.e. solutions of free fields equations on a classical geometry). Quantum gravity effects are non-trivial when one considers self-interacting fields. To illustrate this point, we study the dynamics of a particle, instead of those of a field, in a non-linear potential. We show that the particle is in fact described by a couple of functions $(\Phi_+(t), \Phi_-(t))$, the first one describes the motion of the particle and the second one the reverse motion because we have $\Phi_-(t) = \Phi_+(-t)$: we interpret $\Phi_-(t)$ as the motion of a "miror" particle with respect to $\Phi_+(t)$. We find the equations of motion for $\Phi_\pm$, compute their solutions at the first order (in the amplitude of the non-linear potential) and found differences with classical solutions. This is a very nice feature of our toy-model. Let us emphasize that the quantum gravity effects are a consequence of the discretization of space-time. Similar phenomena occur when discretizing a dynamics for numerical purposes for instance and it has been noticed for a long time that the discretization have strong effect on the dynamics. The main novelty in our model is that the discretization is not put by hand, on contrary it is found from fundamental principles. Furthermore, there is a symmetry (quantum) algebra behind our construction. It would be interesting to study in great details the

effects of that discretization in the dynamics of a general field, in particular to see whether the dynamics, when discretized according to these rules, becomes chaotic or not.

Finally, we propose a background independent interpretation of the dynamics of the particle defining in particular an amplitude associated the particle when it is located at a given fuzzy point at a given time. This amplitude can be positive or negative (so it cannot be really interpreted as a propability) and is maximal near the classical trajectory.

Nevertheless, the model is based on three dimensional Euclidean quantum gravity. What about if space-time becomes Lorentzian? and if space-time is four dimensional? The later question is rather difficult to answer but we can try to apply our technique in the LQG background. Indeed, it has been proposed a description of four-dimensional geometry in terms of non-commutative fuzzy space [14]. The former is much easier to deal with because it should be a straightforward generalisation of our construction. However, many differences should occur due to the fact that the momentum space of the particle is still curved but non-compact. Therefore, the quantum background is still expected to be non-commutative but might be no-longer (completely) discrete. This Lorentzian regime certainly deserves to be studied in details.

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